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Seiberg-Witten map for superfields on canonically deformed $N = 1$, $d = 4$ superspace

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Abstract: In this paper we construct Seiberg-Witten maps for superfields on canonically deformed $N = 1$, $d = 4$ Minkowski and euclidean superspace. On Minkowski superspace we show that the Seiberg-Witten map is not compatible with locality, (anti)chirality and supersymmetry at the same time. On euclidean superspace we show that there exists a local, chiral and supersymmetric Seiberg-Witten map for chiral superfields if we take the noncommutativity parameter to be selfdual, and a local, antichiral and supersymmetric Seiberg-Witten map for antichiral superfields if we take the noncommutativity parameter to be antiselfdual, respectively.

Keywords: Gauge Symmetry, Superspaces, Non-Commutative Geometry.

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1. Introduction

Field theories on canonically deformed space

$$[\hat{x}^i, \hat{x}^j] = i\vartheta^{ij}\,, \qquad \vartheta^{ij} = -\vartheta^{ji} \in \mathbb{R}\,, \tag{1.1}$$

have recently attracted much attention (for reviews and an exhaustive list of references see [1, 2, 3]), mainly due to the discovery of this noncommutative space in string theory [4, 5, 6]. Based on the existence of different regularization procedures in string theory, Seiberg and Witten claimed in [6] that certain noncommutative gauge theories are equivalent to commutative ones. In particular, they argued that there exists a map from a commutative gauge field to a noncommutative one, which is compatible with the gauge structure of each. This map has become known as the Seiberg-Witten map.

In [7]–[10] gauge theory on noncommutative space was formulated using the Seiberg-Witten map. In contrast to earlier approaches [11]–[14], this method works for arbitrary

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gauge groups. Using this method the problems of charge quantisation [15, 16] and tensor product of gauge groups [14] were solved and the standard model and GUT's were formulated at the tree level on noncommutative space [17, 18].

Non(anti)commutative superspaces naturally arise in string theory as well with $x-x$ deformation (canonical deformation) [19], $\theta-\theta$ deformation [20, 21, 22] and $x-\theta$ deformation [23]. General deformed superspaces were first studied more closely in [24, 25] and recently in [26] and in connection with the supermatrix model in [27]. Various aspects of field theory were considered mainly on the canonically deformed superspace and since the work of Seiberg [22] also on the $\theta-\theta$ deformed superspace with $N=1/2$ supersymmetry. On both spaces gauge theory was constructed without Seiberg-Witten map in [24, 28] and [22, 29, 30]. Similar to the nonsupersymetric case, without the Seiberg-Witten map only gauge theories for the $\mathrm{U}(N)$ gauge group can be formulated on these spaces [28, 29].

On the canonically deformed superspace, gauge theory with Seiberg-Witten maps for component fields was considered in [31, 32]. Seiberg-Witten map for superfields was briefly considered in [24, 33]. In [34] we will present the Seiberg-Witten map for superfields on the $\theta-\theta$ deformed superspace.

In this paper we will construct Seiberg-Witten map for superfields on canonically deformed $N=1$, $d=4$ superspace. First we will recapitulate some well known properties of this space and construct the gauge theory coupled to matter without and with Seiberg-Witten map for component fields. Thereafter we will show that on Minkowski space the Seiberg-Witten map is not compatible with the requirements of locality, (anti)chirality and supersymmetry at the same time. We will present Seiberg-Witten maps for superfields in each case where one of these requirements is given up and construct the U(1) gauge theory coupled to matter in terms of classical component fields using the nonsupersymmetric Seiberg-Witten map. Finally we show that on euclidean space there exists a local, chiral and supersymmetric Seiberg-Witten map for chiral fields if ϑ^{ij} is selfdual

$$\vartheta_{ij}=\frac{1}{2}\varepsilon_{ijmn}\vartheta^{mn}\,, \tag{1.2}$$

and a local, antichiral and supersymmetric one if the noncommutativity parameter is antiselfdual

$$\vartheta_{ij}=-\frac{1}{2}\varepsilon_{ijmn}\vartheta^{mn}\,. \tag{1.3}$$

We use the conventions of [35].

2. Canonically deformed superspace

The canonically deformed real $N=1$, $d=4$ superspace, which we denote by $\hat{\mathbb{R}}^{4|4}$, has the following coordinate algebra [19]:

$$\hat{\mathcal{R}}:\begin{cases} [\hat{x}^i,\hat{x}^j]=i\vartheta^{ij}\,,\\ [\hat{x}^i,\hat{\theta}^\alpha]=[\hat{x}^i,\bar{\hat{\theta}}_{\dot\alpha}]=0\,,\\ \{\hat{\theta}_\alpha,\hat{\theta}_\beta\}=\{\bar{\hat{\theta}}_{\dot\alpha},\bar{\hat{\theta}}_{\dot\beta}\}=\{\hat{\theta}_\alpha,\bar{\hat{\theta}}_{\dot\alpha}\}=0\,,\end{cases} \tag{2.1}$$

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with $\vartheta^{ij} \in \mathbb{R}$. Noncommutativity is indicated by a hat. Since the Grassmann coordinates remain classical, in what follows "commutative" and "noncommutative" thus refers only to the bosonic part of coordinate algebra (2.1). Noncommutative functions and fields are defined as elements of the noncommutative algebra

$$\hat{\mathcal{A}} = \frac{\mathbb{C}[[\hat{x}^i, \hat{\theta}_\alpha, \hat{\bar{\theta}}_{\dot{\alpha}}]]}{I_{\hat{\mathcal{R}}}}\,, \tag{2.2}$$

where $I_{\hat{\mathcal{R}}}$ is the two-sided ideal created by the relations (2.1).

The derivatives can be introduced on a noncommutative space in a purely algebraic way [36]. In the case of canonical deformation, both the bosonic and fermionic derivatives act on the coordinates as in the classical case

$$[\hat{\partial}_i, \hat{x}^j] = \delta_i^{\ j}\,, \qquad \{\hat{\partial}_\alpha, \hat{\theta}^\beta\} = \delta_\alpha^{\ \beta}\,, \qquad \dots \tag{2.3}$$

When ϑ^{ij} is invertible, the bosonic derivatives are internal operations in the algebra $\hat{\mathcal{A}}$ and they are given by

$$\hat{\partial}_i \hat{F}(\hat{x}, \hat{\theta}, \hat{\bar{\theta}}) = -i\vartheta^{-1}_{ij}[\hat{x}^j, \hat{F}(\hat{x}, \hat{\theta}, \hat{\bar{\theta}})]\,, \tag{2.4}$$

which leads to the classical relation

$$[\hat{\partial}_i, \hat{\partial}_j] = 0\,. \tag{2.5}$$

It is convenient to use the star product formulation of the algebra. The star product on this noncommutative superspace is the well known Moyal-Wayl star product

$$\begin{aligned}\hat{F}(x,\theta,\bar{\theta}) * \hat{G}(x,\theta,\bar{\theta}) &= e^{\frac{i}{2}\vartheta^{ij}\frac{\partial}{\partial x^i}\frac{\partial}{\partial y^j}}\hat{F}(x,\theta,\bar{\theta})\hat{G}(y,\theta,\bar{\theta})|_{y\to x} \\ &= \hat{F}\hat{G} + \frac{i}{2}\vartheta^{ij}\partial_i\hat{F}\,\partial_j\hat{G} - \frac{1}{8}\vartheta^{ij}\vartheta^{kl}\partial_i\partial_k\hat{F}\partial_j\partial_l\hat{G} + \cdots\,.\end{aligned} \tag{2.6}$$

Some useful properties of this product are:

- involution
$$\overline{\hat{F} * \hat{G}} = \hat{\bar{G}} * \hat{\bar{F}}\,, \tag{2.7}$$

- Leibnitz rule
$$\partial_A(\hat{F} * \hat{G}) = \partial_A\hat{F} * \hat{G} + \hat{F} * \partial_A\hat{G}\,, \tag{2.8}$$
where $\partial_A = (\partial_m, \partial_\alpha, \bar{\partial}_{\dot{\alpha}})$,

- ordinary product of two functions under the integral[1]
$$\int d^8z\, \hat{F} * \hat{G} = \int d^8z\, \hat{F}\hat{G}\,, \tag{2.9}$$
if the functions vanish at infinity (which we demand),

- cyclicity of the product under the integral sign
$$\int d^8z\, \hat{F} * \hat{G} = \int d^8z\, \hat{G} * \hat{F}\,. \tag{2.10}$$

[1] $d^8z = d^4x d^2\theta d^2\bar{\theta}$.

2.1 Symmetries

The algebra (2.1) is covariant under the group of classical supertranslations parametrized by $(\hat{a}, \hat{\xi}, \bar{\hat{\xi}})$

$$
\begin{aligned}
\hat{x}'^m &= \hat{x}^m + \hat{a}^m + i\hat{\theta}\sigma^m\bar{\hat{\xi}} - i\hat{\xi}\sigma^m\bar{\hat{\theta}}\,, \\
\hat{\theta}'_\alpha &= \hat{\theta}_\alpha + \hat{\xi}_\alpha\,, \\
\bar{\hat{\theta}}'_{\dot\alpha} &= \bar{\hat{\theta}}_{\dot\alpha} + \bar{\hat{\xi}}_{\dot\alpha}\,,
\end{aligned}
\tag{2.11}
$$

which is generated by the complex charges $\hat{Q}_\alpha$, $\bar{\hat{Q}}_{\dot\alpha}$ and the four momentum $\hat{P}_m$. The generators have the same representation as in the classical case

$$\hat{P}_m = i\hat{\partial}_m\,, \tag{2.12}$$

$$\hat{Q}_\alpha = \hat{\partial}_\alpha - i(\sigma^m\bar{\hat{\theta}})_\alpha\hat{\partial}_m\,, \tag{2.13}$$

$$\bar{\hat{Q}}_{\dot\alpha} = -\bar{\hat{\partial}}_{\dot\alpha} + i(\hat{\theta}\sigma^m)_{\dot\alpha}\hat{\partial}_m\,. \tag{2.14}$$

The supersymmetry algebra remains undeformed

$$
\begin{aligned}
[\hat{P}_m, \hat{P}_n] &= 0\,, \\
[\hat{P}_m, \hat{Q}_\alpha] &= [\hat{P}_m, \bar{\hat{Q}}_{\dot\alpha}] = 0\,, \\
\{\hat{Q}_\alpha, \hat{Q}_\beta\} &= \{\bar{\hat{Q}}_{\dot\alpha}, \bar{\hat{Q}}_{\dot\beta}\} = 0\,, \\
\{\hat{Q}_\alpha, \bar{\hat{Q}}_{\dot\beta}\} &= 2\,\sigma^m_{\alpha\dot\beta}\,\hat{P}_m\,,
\end{aligned}
\tag{2.15}
$$

because the algebra of derivatives is the same as in commutative case.

2.2 Superfields

The covariant derivatives also have the standard form

$$\hat{D}_\alpha = \hat{\partial}_\alpha + i(\sigma^m\bar{\hat{\theta}})_\alpha\hat{\partial}_m\,, \tag{2.16}$$

$$\bar{\hat{D}}_{\dot\alpha} = -\bar{\hat{\partial}}_{\dot\alpha} - i(\hat{\theta}\sigma^m)_{\dot\alpha}\hat{\partial}_m\,, \tag{2.17}$$

and we can define chiral superfields by $\bar{\hat{D}}_{\dot\alpha}\hat{\Phi} = 0$, antichiral superfields by $\hat{D}_\alpha\bar{\hat{\Phi}} = 0$ and vector superfields by $\hat{V} = \bar{\hat{V}}$. The component expansion is the same as in the classical theory but with noncommutative fields as components

$$\hat{\Phi}(\hat{y}, \hat{\theta}) = \hat{A}(\hat{y}) + \sqrt{2}\hat{\theta}\hat{\psi}(\hat{y}) + \hat{\theta}\hat{\theta}\hat{F}(\hat{y})\,, \tag{2.18}$$

$$\hat{V}(\hat{x}, \hat{\theta}, \bar{\hat{\theta}}) = -\hat{\theta}\sigma^m\bar{\hat{\theta}}\hat{v}_m(\hat{x}) + i\hat{\theta}\hat{\theta}\bar{\hat{\theta}}\bar{\hat{\lambda}}(\hat{x}) - i\bar{\hat{\theta}}\bar{\hat{\theta}}\hat{\theta}\hat{\lambda}(\hat{x}) + \frac{1}{2}\hat{\theta}\hat{\theta}\bar{\hat{\theta}}\bar{\hat{\theta}}\hat{d}(\hat{x})\,, \tag{2.19}$$

where $\hat{y}^m = \hat{x}^m + i\hat{\theta}\sigma^m\bar{\hat{\theta}}$ and $\hat{V}$ is in the Wess-Zumino gauge.

The supersymmetry transformation of the component fields may be found from $\hat{\delta}_\xi\hat{\Phi}$ and $\hat{\delta}_\xi\hat{V}$ where

$$\hat{\delta}_\xi \equiv \left(\hat{\xi}\hat{Q} + \bar{\hat{\xi}}\bar{\hat{Q}}\right). \tag{2.20}$$

Since the generators $\hat{Q}$ and $\bar{\hat{Q}}$ act on the superfields as in the classical case, the transformation of the component fields maintain the classical form as well

$$\hat{\delta}_{\xi}\hat{A} = \sqrt{2}\xi\hat{\psi}\,, \tag{2.21}$$

$$\hat{\delta}_{\xi}\hat{\psi} = i\sqrt{2}\sigma^m\bar{\xi}\hat{\partial}_m\hat{A} + \sqrt{2}\xi\hat{F}\,, \tag{2.22}$$

$$\hat{\delta}_{\xi}\hat{F} = i\sqrt{2}\bar{\xi}\bar{\sigma}^m\hat{\partial}_m\hat{\psi}\,, \tag{2.23}$$

$$\hat{\delta}_{\xi}\hat{v}_m = i\xi\sigma_m\bar{\hat{\lambda}} - i\hat{\lambda}\sigma_m\bar{\xi}\,, \tag{2.24}$$

$$\hat{\delta}_{\xi}\hat{\lambda} = i\xi\hat{d} + 2\sigma^{mn}\xi\hat{\partial}_m\hat{v}_n\,, \tag{2.25}$$

$$\hat{\delta}_{\xi}\hat{d} = \bar{\xi}\bar{\sigma}^m\hat{\partial}_m\hat{\lambda} - \xi\sigma^m\hat{\partial}_m\bar{\hat{\lambda}}\,. \tag{2.26}$$

2.3 Gauge theory

Consider the noncommutative gauge transformation of a chiral superfieldfield $\hat{\Phi}$

$$\hat{\delta}_{\hat{\Lambda}}\hat{\Phi} = -i\hat{\Lambda} * \hat{\Phi}\,, \tag{2.27}$$

with the Lie algebra valued noncommutative gauge parameter $\hat{\Lambda} = \hat{\Lambda}_a T^a$ and $\bar{\hat{D}}\hat{\Lambda} = 0$ in order to preserve chirality. T^a are generators of the appropriate gauge group and form the Lie algebra

$$[T^a, T^b] = if^{ab}_{\ \ c}T^c\,. \tag{2.28}$$

The commutator of two gauge transformations has the same form as in the classical case

$$\hat{\delta}_{\hat{\Lambda}}\hat{\delta}_{\hat{\Sigma}} - \hat{\delta}_{\hat{\Sigma}}\hat{\delta}_{\hat{\Lambda}} = \hat{\delta}_{i[\hat{\Lambda} \stackrel{*}{,} \hat{\Sigma}]}\,, \tag{2.29}$$

but the commutator

$$[\hat{\Lambda} \stackrel{*}{,} \hat{\Sigma}] = \frac{1}{2}\{\hat{\Lambda}_a \stackrel{*}{,} \hat{\Sigma}_b\}[T^a, T^b] + \frac{1}{2}[\hat{\Lambda}_a \stackrel{*}{,} \hat{\Sigma}_b]\{T^a, T^b\} \tag{2.30}$$

only closes into the Lie algebra if the gauge group under consideration is U(N). Thus in this setting gauge theories with gauge groups SU(N) can not be considered. However, using Seiberg-Witten map we can consider SU(N) or arbitrary groups.

In what follows we will restrict our considerations to the gauge group U(1). The generalisation to U(N) is straightforward. The noncommutative supersymmetric U(1) gauge theory coupled to matter, which is the supersymmetric extension of noncommutative electrodynamics, is constructed in terms of two chiral superfields:

$$\hat{\delta}_{\hat{\Lambda}}\hat{\Phi}_+ = -i\hat{\Lambda} * \hat{\Phi}_+\,, \qquad \hat{\delta}_{\hat{\Lambda}}\hat{\Phi}_- = i\hat{\Phi}_- * \hat{\Lambda}\,. \tag{2.31}$$

The action is

$$\hat{\mathcal{S}} = \hat{\mathcal{S}}_{YM} + \mathcal{S}_{int_{(+)}} + \mathcal{S}_{int_{(-)}} + \mathcal{S}_m\,, \tag{2.32}$$

where

$$\hat{\mathcal{S}}_{YM} = \frac{1}{4}\int d^4x d^2\theta\,\left(\hat{W}^\alpha * \hat{W}_\alpha\right) + \text{c.c.}\,, \tag{2.33}$$

$$\hat{\mathcal{S}}_{int_{(+)}} = \int d^4x d^2\theta d^2\bar{\theta} \left(\bar{\hat{\Phi}}_+ * e_*^{g\hat{V}} * \hat{\Phi}_+ \right), \tag{2.34}$$

$$\hat{\mathcal{S}}_{int_{(-)}} = \int d^4x d^2\theta d^2\bar{\theta} \left(\hat{\Phi}_- * e_*^{-g\hat{V}} * \bar{\hat{\Phi}}_- \right), \tag{2.35}$$

$$\hat{\mathcal{S}}_m = m \int d^4x d^2\theta \left(\hat{\Phi}_- * \hat{\Phi}_+ \right) + \text{c.c.}\,. \tag{2.36}$$

The noncommutative exponential function is defined as

$$e_*^{\hat{F}} = \sum_{n=0}^{\infty} \frac{1}{n!} (\hat{F})_*^n = 1 + \hat{F} + \frac{1}{2} \hat{F} * \hat{F} + \frac{1}{6} \hat{F} * \hat{F} * \hat{F} + \cdots . \tag{2.37}$$

Because of the star product noncommutativity, the gauge transformation of the gauge field $\hat{V}$ has the same form as in the nonabelean case

$$\hat{\delta}_{\hat{\Lambda}} \hat{V} = -i\bar{\hat{\Lambda}} * e_*^{\hat{V}} + ie_*^{\hat{V}} * \hat{\Lambda}\,. \tag{2.38}$$

The noncommutative supersymmetric chiral field strength

$$\hat{W}_\alpha = -\frac{1}{4} \bar{\hat{D}} \bar{\hat{D}}\; e_*^{-\hat{V}} * \hat{D}_\alpha * e_*^{\hat{V}} \tag{2.39}$$

has the gauge transformation

$$\hat{\delta}_{\hat{\Lambda}} \hat{W}_\alpha = i[\hat{W} \stackrel{*}{,} \hat{\Lambda}]\,. \tag{2.40}$$

The expansion of the lagrangian in component fields in Wess-Zumino gauge has the same form as the classical nonabelian supersymmetric gauge theory, e.g

$$\mathcal{S}_{YM} = \int d^4x \left(-\frac{1}{4} \hat{f}^{mn} * \hat{f}_{mn} - i\bar{\hat{\lambda}} * \bar{\sigma}^m \hat{\mathcal{D}}_m * \hat{\lambda} + \frac{1}{2} \hat{d} * \hat{d} \right), \tag{2.41}$$

where

$$\hat{f}_{mn} = \hat{\partial}_m \hat{v}_n - \hat{\partial}_n \hat{v}_m + \frac{i}{2} [\hat{v}_m \stackrel{*}{,} \hat{v}_n]\,, \tag{2.42}$$

$$\mathcal{D}_m \hat{\lambda} = \hat{\partial}_m \hat{\lambda} + \frac{i}{2} [\hat{v}_m \stackrel{*}{,} \hat{\lambda}]\,. \tag{2.43}$$

3. Construction of the Seiberg-Witten map in terms of component fields

To determine the Seiberg-Witten map in the supersymmetric case we will start with the supersymmetric Seiberg-Witten equation for the gauge field $\hat{V}$ and the chiral (matter) field $\hat{\Phi}$

$$\hat{V}(V) + \hat{\delta}_{\hat{\Lambda}} \hat{V}(V) = \hat{V}(V + \delta_\Lambda V)\,, \tag{3.1}$$

$$\hat{\Phi}(\Phi, V) + \hat{\delta}_{\hat{\Lambda}} \hat{\Phi}(\Phi, V) = \hat{\Phi}(\Psi + \delta_\Lambda \Phi, V + \delta_\Lambda V)\,, \tag{3.2}$$

To simplify matters we consider the abelian case, since the generalization to nonabelian case is straightforward. We choose furthermore the Wess-Zumino gauge in which the equation (2.38) and the gauge parameter $\hat{\Lambda}$ have the form

$$\hat{\delta}_{\hat{\Lambda}} \hat{V} = i(\hat{\Lambda} - \bar{\hat{\Lambda}}) + \frac{i}{2} [\hat{V} \stackrel{*}{,} \hat{\Lambda} + \bar{\hat{\Lambda}}]\,. \tag{3.3}$$

$$\hat{\Lambda}(y, \theta) = -\hat{\alpha}(y)\,, \tag{3.4}$$

where $\hat{\alpha}$ is the ordinary abelian noncommutative gauge parameter.

For the solution of the Seiberg-Witten equations we use the same procedure as in the nonsupersymmetric case [6, 37]. We expand the superfields in the noncommutative parameter ϑ^{ij}

$$\hat{V} = V + V^{'}(V,\vartheta) + o(\vartheta^2)\,, \tag{3.5}$$
$$\hat{\Phi} = \Phi + \Phi^{'}(\Phi, V,\vartheta) + o(\vartheta^2)\,, \tag{3.6}$$
$$\hat{\Lambda} = \Lambda + \Lambda^{'}(\Lambda, V,\vartheta) + o(\vartheta^2)\,, \tag{3.7}$$

and solve the Seiberg-Witten equations (3.1) and (3.2) perturbatively order by order in the noncommutativity parameter ϑ^{ij}. To zeroth order we get the classical gauge transformations. To first order we get

$$V^{'} - V^{'}(V + \delta_\Lambda V) + i(\Lambda^{'} - \bar{\Lambda}^{'}) = \frac{1}{2}\vartheta^{ij}\,\partial_i V \partial_j(\Lambda + \bar{\Lambda})\,, \tag{3.8}$$
$$\Phi^{'} - \Phi^{'}(\Phi + \delta_\Lambda\Phi, V + \delta_\Lambda V) - i(\Lambda^{'}\Phi + \Lambda\Phi^{'}) = -\frac{1}{2}\vartheta^{ij}\,\partial_i\Lambda\partial_j\Phi\,. \tag{3.9}$$

These equations can be solved easily componentwise. With the assumption that the noncommutative component fields depend only on their classical counterparts and the classical gauge field v_m, we get the following Seiberg-Witten equations for component fields to first order in ϑ^{ij}

$$v_m' - v_m'(v + \delta_\alpha v) - 2\,\partial_m\alpha' = \vartheta^{ij}\,\partial_i\alpha\partial_j v_m\,, \tag{3.10}$$
$$\lambda' - \lambda'(\lambda + \delta_\alpha\lambda, v + \delta_\alpha v) = \vartheta^{ij}\,\partial_i\alpha\partial_j\lambda\,, \tag{3.11}$$
$$d' - d'(d + \delta_\alpha d, v + \delta_\alpha v) = \vartheta^{ij}\,\partial_i\alpha\partial_j d\,, \tag{3.12}$$
$$A' - A'(A + \delta_\alpha A, v_m + \delta_\alpha v_m) + i\alpha' A + i\alpha A' = \frac{1}{2}\vartheta^{ij}\partial_i\alpha\partial_j A\,, \tag{3.13}$$
$$\psi' - \psi'(\psi + \delta_\alpha\psi, v_m + \delta_\alpha v_m) + i\alpha' A + i\alpha A' = \frac{1}{2}\vartheta^{ij}\partial_i\alpha\partial_j\psi\,, \tag{3.14}$$
$$F' - F'(F + \delta_\alpha F, v_m + \delta_\alpha v_m) + i\alpha' F + i\alpha F' = \frac{1}{2}\vartheta^{ij}\partial_i\alpha\partial_j F\,, \tag{3.15}$$

The solution is[2]

$$\alpha'(\alpha, v) = \frac{1}{4}\vartheta^{ij} v_i\partial_j\alpha\,, \tag{3.16}$$
$$v_m'(v) = \frac{1}{4}\vartheta^{ij} v_i(\partial_j v_m + f_{jm})\,, \tag{3.17}$$
$$\lambda'(\lambda, v) = \frac{1}{2}\vartheta^{ij} v_i\partial_j\lambda\,, \tag{3.18}$$
$$d'(d, v) = \frac{1}{2}\vartheta^{ij} v_i\partial_j d\,, \tag{3.19}$$
$$A'(A, v) = \frac{1}{4}\vartheta^{ij} v_i\partial_j A\,, \tag{3.20}$$

[2]We remind that in Wess-Bagger convention, which we use, the gauge transformation of the abelian gauge field v_m has the form $\delta_\alpha v_m = -2\,\partial_m\alpha$. This differs from the usual gauge transformation by the factor -2. For this reason the Seiberg-Witten maps for the fields α', v_m' and ψ' differ from the usual ones [6, 37, 10] by the factor $-\frac{1}{2}$.

$$\psi'(\psi, v) = \frac{1}{4}\vartheta^{ij} v_i \partial_j \psi \,, \tag{3.21}$$

$$F'(F, v) = \frac{1}{4}\vartheta^{ij} v_i \partial_j F \,. \tag{3.22}$$

The nonabelian generalisation of the solutions (3.16)–(3.19) were also found in [31], where the authors have used a different method to determine the Seiberg-Witten equations (3.10)–(3.12).

Using the above expansions one gets the action (2.32) expanded to first order in ϑ^{ij} in terms of classical component fields. For example we get for the linear term in ϑ^{ij} in the Yang-Mills action (2.41)

$$\begin{aligned} \mathcal{S}'_{YM} = \frac{1}{4}\vartheta^{ij} \int d^4x \Big\{ & - f^{mn} f_{mi} f_{nj} + \frac{1}{4} f_{ij} f^{mn} f_{mn} + \\ & + i\big(\partial_i \lambda \sigma^m \bar{\lambda} - \lambda \sigma^m \partial_i \bar{\lambda}\big) f_{jm} + \\ & + \frac{i}{2}\left(\partial_m \lambda \sigma^m \bar{\lambda} - \lambda \sigma^m \partial_m \bar{\lambda} - i d^2\right) f_{ij} \Big\} \,. \end{aligned} \tag{3.23}$$

Due to the construction the ϑ^{ij} expanded action is invariant under commutative gauge transformations. But, as was realised in [31], it is not covariant under the commutative supersymmetry transformation. However, one can also expand the supersymmetry generators in ϑ^{ij} where the expanded terms depend on the commutative component fields of the gauge superfield in such a way that the action will be covariant in each order under the action of these expanded supersymmetry generators. This was outlined in [31, 38] (see also [32]).

The solutions (3.16)–(3.22) are very restrictive because we assumed that the noncommutative component fields depend only on their commutative counterparts and the classical gauge field v_m. Indeed, there is no reason for such a restriction. Giving up this restriction one could ask if it is possible to get solutions which will lead to an action which is invariant under classical supersymmetry. This question is automatically answered by solving equations (3.8) and (3.9) explicitly in terms of superfields.

To solve equations (3.8) and (3.9) we first have to find simultaneously the Seiberg-Witten map for Λ' and V'. To avoid this we will apply the method developed by Wess and collaborators in [7]–[10] to determine the Seiberg-Witten map for the superfield case.

4. Construction of the Seiberg-Witten map in terms of Superfields

We consider again the noncommutative gauge transformation of a chiral matter field (2.27), but with enveloping algebra valued gauge parameter $\hat{\Lambda}$:

$$\hat{\Lambda} = \Lambda_a T^a + \Lambda'_{ab} : T^a T^b : + \Lambda''_{abc} : T^a T^b T^c : + \cdots \,, \tag{4.1}$$

where Λ' is linear in ϑ^{ij}, Λ'' is quadratic in ϑ^{ij} etc. The dots indicate that we have to sum over a basis of the vector space spanned by the homogenous polynomials in the generators T^a of the Lie algebra. Completely symmetrized products could serve as a basis:

$$: T^a := T^a \,, \qquad : T^a T^b := \frac{1}{2}(T^a T^b + T^b T^a) \,, \qquad \text{etc.} \tag{4.2}$$

The commutator of two transformations (2.29) is certainly enveloping algebra valued. Hence we can use arbitrary Lie groups but the price we seem to have to pay is an infinite number of gauge parameters and an infinite number of gauge fields.

To avoid this problem we define new gauge transformations, where all these infinitely many gauge parameters depend just on the classical gauge parameter Λ, the classical gauge field V and on their derivatives. We assume moreover that all superfields considered (e.g. $\hat{\Phi}$, $\hat{V}$) depend on their classical counterpart, the classical gauge field V and on their derivatives. This dependence, which we call Seiberg-Witten map, will be denoted by $\hat{\Lambda}(\Lambda, V)$, $\hat{\Phi}(\Phi, V)$ and $\hat{V}(V)$.

The gauge transformations for the noncommutative chiral matter field and the noncommutative gauge field now have the form

$$\delta_\Lambda \hat{\Phi}(\Phi, V) = -i\hat{\Lambda}(\Lambda, V) * \hat{\Phi}(\Phi, V)\,, \tag{4.3}$$

$$\delta_\Lambda \hat{V}(V) = -i\bar{\hat{\Lambda}}(\bar{\Lambda}, V) * e_*^{\hat{V}(V)} + ie_*^{\hat{V}(V)} * \hat{\Lambda}(\Lambda, V)\,. \tag{4.4}$$

Since equation (2.29), called consistency condition in [10], involves solely the gauge parameters, it is convenient to base the construction of the Seiberg-Witten map on it. In a second step the remaining Seiberg-Witten maps for the matter field and the gauge field can be computed from the equations (4.3) and (4.4).

The procedure in the abelian case is the following. As was mentioned, we start with the consistency condition which has the following form in the abelian case

$$(\delta_\Lambda \delta_\Sigma - \delta_\Sigma \delta_\Lambda)\, \hat{\Phi}(\Phi, V) = 0\,. \tag{4.5}$$

With equation (4.3) we get more explicitly

$$-\, i\delta_\Lambda \hat{\Sigma}(\Sigma, V) + i\delta_\Sigma \hat{\Lambda}(\Lambda, V) - \hat{\Sigma}(\Sigma, V) * \hat{\Lambda}(\Lambda, V) + \hat{\Lambda}(\Lambda, V) * \hat{\Sigma}(\Sigma, V) = 0\,. \tag{4.6}$$

The variation $\delta_\Lambda \hat{\Sigma}(\Sigma, V)$ refers to the V-dependence of $\hat{\Sigma}(\Sigma, V)$ and the gauge transformation of the supersymmetric abelian gauge field V

$$\delta_\Lambda V = i(\Lambda - \bar{\Lambda})\,. \tag{4.7}$$

We now expand the consistency condition and the gauge transformations (4.3) and (4.4) using the expansions (3.5)–(3.7) and the expanded star product (2.6). From these expanded equations we can then determine the Seiberg-Witten maps order by order in ϑ^{ij} for all considered superfields. To first order in ϑ^{ij} we get the following equations

$$\delta_\Lambda \Sigma'(\Sigma, V) - \delta_\Sigma \Lambda'(\Lambda, V) = \vartheta^{ij} \partial_i \Lambda \partial_j \Sigma\,, \tag{4.8}$$

$$\delta_\Lambda \Phi'(\Phi, V) + i\Lambda \Phi'(\Phi, V) + i\Lambda'(\Lambda, V)\Phi = \frac{1}{2}\vartheta^{ij} \partial_i \Lambda \partial_j \Phi\,, \tag{4.9}$$

$$\delta_\Lambda V'(V) - i\Lambda'(\Lambda, V) + i\bar{\Lambda}'(\bar{\Lambda}, V) = \frac{1}{2}\vartheta^{ij} \partial_i (\Lambda + \bar{\Lambda}) \partial_j V\,. \tag{4.10}$$

We will now look for solutions of these equations.

5. Seiberg-Witten map on Minkowski space

It is reasonable to require the following properties of the Seiberg-Witten map for superfields:

1. locality (in each order),
2. chirality of the gauge parameter (otherwise we would not have a gauge theory which involves only one real vector superfield $\hat{V}$ since it would not be possible to impose the so called *representation preserving constraints* [39, 40]),
3. covariance under classical supersymmetry.

First we will show that on canonically deformed Minkowski superspace there is no Seiberg-Witten map for $\Lambda^{'}$ which fulfils all three requirements. However, as we will see, there are solutions if we give up one of the requirements.

5.1 No go theorem

Theorem. *On canonically deformed Minkowski superspace (see equation (2.1)) there is no Seiberg-Witten map for the gauge parameter $\hat{\Lambda}$ which is local, chiral and supersymmetric at the same time.*

To prove this theorem, it is sufficient to prove that there is no local, chiral and supersymmetric solution of the consistency condition to first order in ϑ^{ij} (4.8). We show this by using dimensional analysis.

The right hand side of the consistency condition (4.8) is linear in each of the classical superfields Λ and Σ. All terms in the ansatz for $\Lambda^{'}$ which would contain powers of V can therefore solve only the homogeneous consistency condition because of (4.7). Hence we make an ansatz for $\Lambda^{'}$ only linear in V without loss of generality. Moreover $\Lambda^{'}$ has to be linear in the classical gauge parameter Λ and by definition linear in ϑ^{ij}. In order to have a supersymmetric expression we may use the bosonic derivatives ∂_m and the covariant derivatives D_α and $\bar{D}_{\dot{\alpha}}$ only. The mass dimensions of these objects are

$$[\Lambda^{'}] = [\Lambda] = [V] = 0\,, \qquad [\vartheta^{ij}] = -2\,, \qquad [\partial_m] = 1\,, \qquad [D_\alpha] = [\bar{D}_{\dot{\alpha}}] = \frac{1}{2}\,. \tag{5.1}$$

It is not hard to see that there is only one local, chiral and supersymmetric combination of ϑ^{ij}, Λ, V, ∂_m, D_α and $\bar{D}_{\dot{\alpha}}$ with appropriate index contraction and mass dimension zero. It is [24]

$$\Lambda^{'} = aP^{\alpha\beta}\bar{D}^2(D_\alpha\Lambda D_\beta V)\,, \tag{5.2}$$

where a is a constant and

$$P^{\alpha\beta} = \vartheta_{ij}\varepsilon^{\alpha\mu}(\sigma^{ij})_\mu{}^\beta\,. \tag{5.3}$$

We put this ansatz for $\Lambda^{'}$ and $\Sigma^{'}$ in the left hand side of the consistency condition (4.8) and get

$$\begin{aligned}\delta_\Lambda\Sigma^{'} - \delta_\Sigma\Lambda^{'} &= 32aiTr\left(\sigma^{ij}\sigma^{mn}\right)\partial_m\Lambda\partial_n\Sigma \\ &= -32ai\left(\vartheta^{ij} + \frac{i}{2}\varepsilon^{ijmn}\vartheta_{mn}\right)\partial_i\Lambda\partial_j\Sigma\,.\end{aligned} \tag{5.4}$$

There is no choice of the constant a which makes this expression equal to the right hand side of the consistency condition. Thus we proved the above theorem.

Nevertheless, it is obvious from equation (5.4) that the ansatz (5.2) would be a solution of the consistency condition if we would take ϑ^{ij} to be complex and selfdual. But in this case we would have to complexify the canonically deformed Minkowski superspace. The other possibility is to consider the canonically deformed euclidean superspace on which the selfdual ϑ^{ij} is real. Before we do this, let us consider the Seiberg-Witten maps for superfields on deformed Minkowski superspace when one of the above requirements is given up.

5.2 Nonchiral solution

A nonchiral solution of the consistency condition (4.8) is

$$\Lambda'(\Lambda, V) = -\frac{1}{16} P^{\dot\alpha\alpha\dot\beta\beta}(\partial_{\alpha\dot\alpha}\Lambda)(\bar D_{\dot\beta} D_\beta V)\,, \tag{5.5}$$

where $P^{\dot\alpha\alpha\dot\beta\beta}$ and $\partial_{\alpha\dot\alpha}$ are the noncommutative parameter and the bosonic derivative in spinor notation

$$P^{\dot\alpha\alpha\dot\beta\beta} = \vartheta_{ij}\bar\sigma^{i\,\dot\alpha\alpha}\bar\sigma^{j\,\dot\beta\beta}\,, \qquad \partial_{\alpha\dot\alpha} = \sigma^m_{\alpha\dot\alpha}\partial_m\,. \tag{5.6}$$

One gets this nonchiral Seiberg-Witten map for the gauge parameter immediately by the attempt to write the equation

$$\Lambda'\big|_{\theta=\bar\theta=0} = -\alpha' = -\frac{1}{4}\vartheta^{ij} v_i \partial_j \alpha \tag{5.7}$$

in a supersymmetric way in terms of the superfields Λ and V since

$$\begin{aligned} \partial_{\alpha\dot\alpha}\Lambda\big|_{\theta=\bar\theta=0} &= -\sigma^m_{\alpha\dot\alpha}\partial_m \alpha\,, && (5.8)\\ \bar D_{\dot\beta} D_\beta V\big|_{\theta=\bar\theta=0} &= \sigma^m_{\beta\dot\beta}\, v_m\,. && (5.9)\end{aligned}$$

With this solution we can now solve the equations (4.9) and (4.10) and obtain the following Seiberg-Witten maps for Φ' and V':

$$\begin{aligned} \Phi'(\Phi, V) &= -\frac{1}{16} P^{\dot\alpha\alpha\dot\beta\beta}(\partial_{\alpha\dot\alpha}\Phi)(\bar D_{\dot\beta} D_\beta V)\,, && (5.10)\\ V'(V) &= -\frac{1}{16} P^{\dot\alpha\alpha\dot\beta\beta}(\partial_{\alpha\dot\alpha}V)(\bar D_{\dot\beta} D_\beta V)\,. && (5.11)\end{aligned}$$

The solution (5.10) is nonchiral as well.

In the Wess-Zumino gauge we get the following component expansion of these solutions:

$$\begin{aligned} \Lambda'(y,\theta,\bar\theta) &= -\frac{1}{4}\vartheta^{ij}\Big\{ v_i\partial_j\alpha + i\theta\sigma_i\bar\lambda\partial_j\alpha + i\bar\theta\bar\sigma_i\lambda\partial_j\alpha - i\theta\sigma^m\bar\theta\left[\left(f^{SD}_{jm} - i\eta_{mj}d\right)\partial_i\alpha\right] + \\ &\qquad + \theta\theta\bar\theta\bar\sigma_j\sigma^m\partial_m\bar\lambda\partial_i\alpha\Big\}\,, \qquad (5.12)\\ \Phi(y,\theta,\bar\theta) &= -\frac{1}{4}\vartheta^{ij}\Big\{\partial_i A v_j + \sqrt{2}\theta\left(\partial_i\psi v_j + \frac{i}{\sqrt{2}}\sigma_j\bar\lambda\partial_i A\right) + \\ &\qquad + i\bar\theta\bar\sigma_j\lambda\partial_i A + \theta\theta\left[\partial_i F v_j - \frac{i}{\sqrt{2}}\partial_i\psi\sigma_j\bar\lambda\right] + \end{aligned}$$

$$
\begin{aligned}
&+ i\theta\sigma^m\bar\theta\Big[\Big(f^{SD}_{jm} - i\eta_{mj}d\Big)\partial_i A - \frac{i}{\sqrt{2}}\lambda\sigma_j\bar\sigma_m\partial_i\psi\Big] + \qquad (5.13)\\
&+ \theta\theta\bar\theta\Big[\frac{i}{\sqrt{2}}\Big(\bar\sigma^m f^{SD}_{jm} - i\bar\sigma_j d\Big)\partial_i\psi + i\bar\sigma_j\lambda\partial_i F - \bar\sigma_j\sigma^m\partial_m\bar\lambda\partial_i A\Big]\Big\}\,,
\end{aligned}
$$

$$
\begin{aligned}
V^{'}(x,\theta,\bar\theta) = \frac{1}{4}\vartheta^{ij}\Big\{&\theta\sigma^m\bar\theta\,(v_j\partial_i v_m) - i\theta\theta\bar\theta\left[\partial_i\bar\lambda v_j - \frac{1}{2}\bar\sigma^m\sigma_j\bar\lambda\partial_i v_m\right] + \\
&+ i\bar\theta\bar\theta\theta\left[\partial_i\lambda v_j - \frac{1}{2}\sigma^m\bar\sigma_j\lambda\partial_i v_m\right] - \\
&- \frac{1}{2}\theta\theta\bar\theta\bar\theta\left[\partial_i(v_j d) - \partial_i v^m\, {}^\star f_{jm} - \partial_i(\lambda\sigma_j\bar\lambda)\right]\Big\}\,, \qquad (5.14)
\end{aligned}
$$

where ${}^\star f_{ab} = \frac{1}{2}\varepsilon_{abcd}f^{cd}$ and f^{SD}_{mn} is the complexifyed selfdual field strength defined as usual: $f^{SD}_{mn} = f_{mn} + i\,{}^\star f_{mn}$.

From the component expansion we see that there are additional peculiar aspects of this solution besides the nochirality of the gauge parameter. From (5.14) we read off the Seiberg-Witten map for the ordinary gauge field v'_m

$$
v'_m = \frac{1}{4}\vartheta^{ij}v_i\partial_j v_m\,, \qquad (5.15)
$$

and realise that it does not coincide with the original one of Seiberg and Witten which is (3.17). Furthermore, from the equation (5.12) we see that the $-i\theta\sigma^m\bar\theta$ component of $\Lambda^{'}(x,\theta,\bar\theta)$ is

$$
\Lambda^{'}(x,\theta,\bar\theta)\big|_{-i\theta\sigma^m\bar\theta} = -\partial_m\alpha' + f'_m\,, \qquad (5.16)
$$

where

$$
\begin{aligned}
\alpha'(\alpha,v) &= \frac{1}{4}\vartheta^{ij}v_i\partial_j\alpha\,, &(5.17)\\
f'_m(\alpha,v,d) &= -\frac{1}{4}\vartheta^{ij}\left(f^{SD}_{jm} - i\eta_{mj}d\right)\partial_i\alpha\,. &(5.18)
\end{aligned}
$$

This implements an unusual gauge transformation of the noncommutative abelian gauge field $\hat v_m$

$$
\delta_\alpha\hat v_m = -2\hat\partial_m\hat\alpha + i[\hat\alpha \stackrel{*}{,} \hat v_m] - (\hat f_m + \bar{\hat f}_m)\,. \qquad (5.19)
$$

We want to make one more comment on the nonchirality of the superfield $\hat\Lambda$. Since $\Lambda^{'}$ is a complex linear superfield satisfying the constraint $\bar D^2\Lambda^{'} = 0$, one could ask if the same is true for $\hat\Lambda$. This would be the case if the Seiberg-Witten map to all orders would satisfy this constraint. With some computation one can show that

$$
\Lambda''(\Lambda,V) = -\frac{i}{2^8}P^{\dot\alpha\alpha\dot\beta\beta}P^{\dot\mu\mu\dot\nu\nu}(\bar D_{\dot\alpha}D_\alpha\Lambda)(\bar D_{\dot\mu}D_\mu V)\partial_{\beta\dot\beta}(\bar D_{\dot\nu}D_\nu V) \qquad (5.20)
$$

solves the consistency condition expanded to second order in the noncommutative parameter

$$
\delta_\Lambda\Sigma^{''} - \delta_\Sigma\Lambda^{''} = \vartheta^{ij}\partial_i\Lambda\partial_j\Sigma^{'} - \vartheta^{ij}\partial_i\Sigma\partial_j\Lambda^{'}\,. \qquad (5.21)
$$

This solution does not satisfy the constraint $\bar D^2\Lambda^{''} = 0$ and is therefore not a complex linear superfield. Since this solution is not unique, it is still an open question if there exists a complex linear solution to second order.

5.3 Nonlocal solution

The most natural way to create a chiral Seiberg-Witten map for the supersymmetric gauge parameter out of the nonchiral one (5.5) is to insert the chiral projector

$$P = \frac{\bar{D}^2 D^2}{16\Box}\,, \tag{5.22}$$

where $\Box = \partial^m \partial_m$ in front of the nonchiral term $\bar{D}_{\dot{\alpha}} D_\alpha V$. The Seiberg-Witten map created in this way

$$\Lambda^{'}(\Lambda, V) = -\frac{1}{16} P^{\dot{\alpha}\alpha\dot{\beta}\beta}(\partial_{\alpha\dot{\alpha}}\Lambda) P(\bar{D}_{\dot{\beta}} D_\beta V)\,, \tag{5.23}$$

is still a solution of the consistency condition (4.8).

The nonlocality enters in this solution due to the nonlocal chiral projector. As a matter of fact, the chiral constraint $\bar{D}_{\dot{\alpha}}\Lambda^{'} = 0$ contains derivatives and is like a differential equation in superspace. Hence, it is not ultralocal in the x variables (ultralocality means that the constraint is determined exclusively by the function at x, but not it's derivatives). This is why the chiral projectors on fields are nonlocal. One has to invert derivatives with respect to x.

The corresponding nonlocal solutions of the equations (4.9) and (4.10) are

$$\Phi^{'}(\Phi, V) = -\frac{1}{16} P^{\dot{\alpha}\alpha\dot{\beta}\beta}(\partial_{\alpha\dot{\alpha}}\Phi) P(\bar{D}_{\dot{\beta}} D_\beta V)\,. \tag{5.24}$$

$$\begin{aligned} V^{'}(V) &= -\frac{1}{16} P^{\dot{\alpha}\alpha\dot{\beta}\beta}(\partial_{\alpha\dot{\alpha}}V) P(\bar{D}_{\dot{\beta}} D_\beta V) + \frac{1}{16} P^{\dot{\alpha}\alpha\dot{\beta}\beta}(\partial_{\alpha\dot{\alpha}}V) \bar{P}(D_\beta \bar{D}_{\dot{\beta}} V) - \\ &\quad - \frac{i}{32} P^{\dot{\alpha}\alpha\dot{\beta}\beta}(P\bar{D}_{\dot{\alpha}} D_\alpha V)(\bar{P} D_\beta \bar{D}_{\dot{\beta}} V)\,. \end{aligned} \tag{5.25}$$

It is obvious that $\bar{D}_{\dot{\alpha}}\Phi^{'} = 0$. Since $P\bar{D}_{\dot{\alpha}} D_\alpha = -2iP\partial_{\alpha\dot{\alpha}}$ we can write the Seiberg-Witten maps (5.23)–(5.25) in a simpler form

$$\Lambda^{'}(\Lambda, V) = \frac{1}{2}\vartheta^{ij}\partial_i \Lambda P \partial_j V\,, \tag{5.26}$$

$$\Phi^{'}(\Phi, V) = \frac{1}{2}\vartheta^{ij}\partial_i \Phi P \partial_j V\,, \tag{5.27}$$

$$V^{'}(V) = \frac{i}{2}\vartheta^{ij}(1-P)\partial_i V (1-\bar{P})\partial_j V\,. \tag{5.28}$$

The solutions (5.26) and (5.28) were first found in [33].

The nonlocal Seiberg-Witten maps for the component fields can be deduced from the nonsupersymmetric Seiberg-Witten maps that we will now discuss.

5.4 Nonsupersymmetrtic solution

Another possibility in order to modify equation (5.5) into a chiral one is to use the nonsupersymmetric chiral projector M

$$M = \frac{1}{16\Box}\bar{D}^2 D^\alpha M_\alpha\,, \tag{5.29}$$

where

$$M_\alpha = -\partial_\alpha + 3i\left(\sigma^m\bar\theta\right)_\alpha \partial_m\,, \tag{5.30}$$

instead of the supersymmetric one (5.22). The nonsupersymmetric Seiberg-Witten maps for the fields $\Lambda^{'}$, $\Phi^{'}$ and $V^{'}$ are therefore the same as (5.23)-(5.25), where the chiral projector P is replaced by M and the antichiral projector $\bar P$ by $\bar M$, respectively.

It is worth noticing that Q_α commutes with M and $\bar Q_{\dot\alpha}$ with $\bar M$. The chiral fields $\Lambda^{'}$ and $\Phi^{'}$ retain therefore $N=(1/2,0)$ and the antichiral fields $\bar\Lambda^{'}$ and $\bar\Phi^{'}$ $N=(0,1/2)$ of the original $N=(1/2,1/2)$ supersymmetry.[3] The vector superfield $V^{'}$ however brakes the entire $N=(1/2,1/2)$ supersymmetry because it contains both projectors.

The fields $\Lambda^{'}$ and $V^{'}$ have the following component expansion in the Wess-Zumino gauge

$$\Lambda^{'}(y,\theta) = -\,\alpha'(y) + \theta\eta'(y)\,, \tag{5.31}$$

$$\begin{aligned} V^{'}(x,\theta,\bar\theta) &= i\theta\chi'(x) - i\bar\theta\bar\chi'(x) - \,\theta\sigma^m\bar\theta v'_m(x) + i\theta\theta\bar\theta\left[\bar\lambda'(x) + \frac{i}{2}\bar\sigma^m\partial_m\chi'(x)\right] - \\ &\quad - i\bar\theta\bar\theta\theta\left[\lambda'(x) + \frac{i}{2}\sigma^m\partial_m\bar\chi'(x)\right] + \frac{1}{2}\theta\theta\bar\theta\bar\theta d'(x)\,. \end{aligned} \tag{5.32}$$

These are different from the classical ones (3.4) and (2.19), while the component expansion of $\Phi^{'}$ is still the classical one (2.18). The Seiberg-Witten maps for the component fields are

$$\alpha'(\alpha,v) = -\frac{1}{4}\vartheta^{ij}\partial_i\alpha v_j\,, \tag{5.33}$$

$$\eta'(\alpha,\bar\lambda) = -\frac{i}{4}\vartheta^{ij}\sigma_i\bar\lambda\partial_j\alpha\,, \tag{5.34}$$

$$A^{'}(A,v) = -\frac{1}{4}\vartheta^{ij}\partial_i A v_j\,, \tag{5.35}$$

$$\psi'(\psi,v,\bar\lambda) = -\frac{1}{4}\vartheta^{ij}\left(\partial_i\psi v_j + \frac{i}{\sqrt2}\sigma_j\bar\lambda\partial_i A\right), \tag{5.36}$$

$$F^{'}(F,\psi,v,\bar\lambda) = -\frac{1}{4}\vartheta^{ij}\left(\partial_i F v_j - \frac{i}{\sqrt2}\partial_i\psi\sigma_j\bar\lambda\right), \tag{5.37}$$

$$\chi'(\bar\lambda,v) = \frac{i}{8}\vartheta^{ij}\sigma_i\bar\lambda v_j\,, \tag{5.38}$$

$$v^{'}_m(v,\lambda,\bar\lambda) = \frac{1}{4}\vartheta^{ij}\left(v_i\left(\partial_j v_m + f_{jm}\right) + \frac{1}{4}\varepsilon_{ijmn}\lambda\sigma^n\bar\lambda\right), \tag{5.39}$$

$$\lambda^{'}(\lambda,v) = -\frac{1}{4}\vartheta^{ij}\left(\partial_i\lambda v_j - \frac{1}{2}\sigma^m\bar\sigma_i\lambda f_{mj}\right), \tag{5.40}$$

$$d^{'}(d,v,\lambda,\bar\lambda) = -\frac{1}{4}\vartheta^{ij}\left(2\partial_i d v_j - \partial_i\lambda\sigma_j\bar\lambda + \frac{i}{2}{}^\star(\partial_i\lambda\sigma_j\bar\lambda) - \lambda\sigma_j\partial_i\bar\lambda - \frac{i}{2}{}^\star(\lambda\sigma_j\partial_i\bar\lambda)\right), \tag{5.41}$$

where ${}^\star(\partial_i\lambda\sigma_j\bar\lambda) = \frac{1}{2}\varepsilon_{ijmn}\partial^m\lambda\sigma^n\bar\lambda$.

[3] Recently, Seiberg [22] considered the $\theta-\theta$ deformed superspace which keeps the $N=1/2$ supersymmetry and has some interesting properties from the field theoretical viewpoint (see e.g. [41] and references therein). Therefore it is more suitable to denote the $N=1$ superspace as $N=(1/2,1/2)$ superspace.

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Comparing these solutions with the solutions obtained in section 3, we see that additional terms enter which are solutions of the homogenous Seiberg-Witten equations. Hence, these solutions are connected to the solutions in section 3 via field redefinition. Furthermore, the Seiberg-Witten maps (5.33), (5.36) and (5.39) coincide with the original ones proposed by Seiberg and Witten when the superpartner fields are set to zero.

The expansion of the field strength in classical component fields can be determined from equations (2.39), (5.32) and (5.38)–(5.41). It is

$$W'_\alpha(y,\theta,\bar\theta) = \frac{1}{4}\,\vartheta^{ij} W_{\alpha\, ij}\,, \tag{5.42}$$

where

$$\begin{aligned} W_{\alpha\, ij} \;=\;& 2i\,\partial_i\lambda_\alpha v_j + \frac{i}{2}\left(\sigma^m\bar\sigma_i\lambda\right)_\alpha f_{jm} - \\ & -\theta_\alpha\left[2\partial_i d v_j - \partial_i\lambda\sigma_j\bar\lambda + \frac{i}{2}{}^\star\!\left(\partial_i\lambda\sigma_j\bar\lambda\right) - \lambda\sigma_j\partial_i\bar\lambda - \frac{i}{2}{}^\star\!\left(\lambda\sigma_j\partial_i\bar\lambda\right)\right] + \\ & +2i\left(\sigma^{mn}\theta\right)_\alpha\left[f_{mi}f_{nj} - v_i\partial_j f_{mn} - \frac{i}{8}\,\partial_m(\lambda\sigma_i\bar\sigma_n\sigma_j\bar\lambda) + \frac{i}{8}\,\partial_n\left(\lambda\sigma_i\bar\sigma_m\sigma_j\bar\lambda\right)\right] + \\ & +\theta\theta\left[2\left(\sigma^m\partial_i\bar\lambda\right)_\alpha f_{jm} - 2\left(\sigma^m\partial_m\partial_i\bar\lambda\right)_\alpha v_j + \frac{1}{2}\left(\sigma^m\bar\sigma^n\sigma_i\partial_m\left(\bar\lambda f_{nj}\right)\right)_\alpha\right]. \end{aligned} \tag{5.43}$$

Now we want to make a remark on the Seiberg-Witten map for the component fields of the nonlocal solutions. The difference lies in using different projectors. The difference in the component expansion comes from the terms on which the projectors act. These terms are

$$\left(M\bar D\sigma_j DV\right)(y,\theta) \;=\; -2v_j - 2i\theta\sigma_j\bar\lambda\,, \tag{5.44}$$

$$\left(P\bar D\sigma_j DV\right)(y,\theta) \;=\; -2\left(\frac{\partial_j\partial_m}{\Box}v^m - i\frac{\partial_j}{\Box}d\right) + 4i\theta\sigma^m\frac{\partial_j\partial_m}{\Box}\bar\lambda\,. \tag{5.45}$$

From this we see that we get the Seiberg-Witten map for the component fields of the nonlocal solution by the following replacement of each v_j and $\sigma_j\bar\lambda$ (respectively v_i and $\sigma_i\bar\lambda$) in equations (5.33)–(5.41):

$$v_j \;\longrightarrow\; \frac{\partial_j\partial_m}{\Box}v^m - i\frac{\partial_j}{\Box}d\,, \tag{5.46}$$

$$\sigma_j\bar\lambda \;\longrightarrow\; -2\sigma^m\frac{\partial_j\partial_m}{\Box}\bar\lambda\,. \tag{5.47}$$

5.5 Nonsupersymmetric noncommutative electrodynamics

We are now able using the Seiberg-Witten maps (5.35)–(5.41) to expand the action (2.32) in classical component fields to first order in ϑ^{ij}. With the expansion of the superfields and star product we get

$$\mathcal{S}'_{YM} = \frac{1}{2}\int d^4x d^2\theta\,\left(W'^\alpha W_\alpha\right) + \text{c.c.}\,, \tag{5.48}$$

$$\mathcal{S}'_{int_{(+)}} = \int d^4x d^2\theta d^2\bar\theta\Big\{\bar\Phi_+\Phi'_+ + e\left(\bar\Phi_+ V\Phi'_+ + \frac{i}{4}\vartheta^{ij}\bar\Phi_+\partial_i V\partial_j\Phi_+\right) +$$

$$+\frac{e^2}{2}\left(\bar\Phi_+V^2\Phi'_+ + \bar\Phi_+VV'\Phi_+ + \frac{i}{4}\vartheta^{ij}\bar\Phi_+\partial_iV^2\partial_j\Phi_+\right)\Bigg\} + \text{c.c.}\,, \tag{5.49}$$

$$\mathcal{S}'_m = m\int d^4xd^2\theta\,\left(\Phi'_-\Phi_+ + \Phi_-\Phi'_+\right) + \text{c.c.}\,. \tag{5.50}$$

We omit the action $\mathcal{S}'_{int_{(-)}}$ since it can be easily deduced from the action $\mathcal{S}'_{int_{(+)}}$. The Seiberg-Witten maps for the component fields of the chiral superfields Φ'_+ and Φ'_- have the same form (5.35)–(5.37). The action (2.32) expanded in classical components via Seiberg-Witten map to first order in ϑ^{ij} is

$$\begin{aligned}\mathcal{S}'_{YM} &= \text{Equation (3.23)} + \\ &\quad+\frac{1}{4}\int d^4x\,\vartheta^{ij}\bigg\{\Big[\partial_i\lambda\sigma_j\bar\lambda - \frac{i}{2}{}^\star(\partial_i\lambda\sigma_j\bar\lambda) + \lambda\sigma_j\partial_i\bar\lambda + \frac{i}{2}{}^\star(\lambda\sigma_j\partial_i\bar\lambda)\Big]d \\ &\qquad\qquad + \frac{i}{2}\Big[\lambda\sigma_i\bar\sigma^m\sigma^n\partial_n\bar\lambda - \partial_n\lambda\sigma^n\bar\sigma^m\sigma_i\bar\lambda\Big]f_{jm}\bigg\} \\ &= \frac{1}{4}\int d^4x\,\vartheta^{ij}\bigg\{ - f^{mn}f_{mi}f_{nj} + \frac{1}{4}f_{ij}f^{mn}f_{mn} + \frac{1}{2}d^2f_{ij} + \\ &\qquad + \frac{i}{2}\Big[\partial_i\lambda\sigma^m\bar\lambda + i\,{}^\star(\partial_i\lambda\sigma^m\bar\lambda) - \lambda\sigma^m\partial_i\bar\lambda + i\,{}^\star(\lambda\sigma^m\partial_i\bar\lambda) + \\ &\qquad\quad + \partial^m\lambda\sigma_i\bar\lambda - i\,{}^\star\big(\partial^m\lambda\sigma_i\bar\lambda\big) - \lambda\sigma_i\partial^m\bar\lambda - i\,{}^\star\big(\lambda\sigma_i\partial_m\bar\lambda\big)\Big]f_{jm} + \\ &\qquad + \Big[\partial_i\lambda\sigma_j\bar\lambda - \frac{i}{2}{}^\star(\partial_i\lambda\sigma_j\bar\lambda) + \lambda\sigma_j\partial_i\bar\lambda + \frac{i}{2}{}^\star(\lambda\sigma_j\partial_i\bar\lambda)\Big]d\bigg\}\,,\end{aligned} \tag{5.51}$$

$$\begin{aligned}\mathcal{S}'_{int_{(+)}} &= \frac{1}{4}\int d^4x\,\vartheta^{ij}\bigg\{\partial^m\bar A\partial_m(\partial_iAv_j) + \partial^mA\partial_m(\partial_i\bar Av_j) - \frac{1}{2}F\bar Ff_{ij} + \\ &\quad + i(\partial_i\psi\sigma^m\partial_m\bar\psi - \partial_m\psi\sigma^m\partial_i\bar\psi)v_j + \frac{i}{\sqrt2}(\bar F\partial_i\psi\sigma_j\bar\lambda - F\lambda\sigma_j\partial_i\bar\psi) + \\ &\quad + \frac{1}{\sqrt2}(\partial_m\bar\psi\bar\sigma^m\sigma_j\bar\lambda\partial_iA + \lambda\sigma_j\bar\sigma^m\partial_m\psi\partial_i\bar A) + \\ &\quad + \frac{e}{2}\Big[2(\partial_j\bar A\partial^mA + \partial^m\bar A\partial_jA)\partial_iv_m - 2i\bar A\partial_jA\partial_id + \\ &\qquad + 2\sqrt2\big(\bar A\partial_i\lambda\partial_j\psi + A\partial_i\bar\lambda\partial_j\bar\psi\big) + 2i\partial_iv_m\partial_j\psi\sigma^m\bar\psi + \\ &\qquad + i\big(\bar A\partial_iA - A\partial_i\bar A\big)v_j\partial^mv_m - \big(\bar A\partial_iA + A\partial_i\bar A\big)\big(dv_j - \lambda\sigma_j\bar\lambda\big) + \\ &\qquad + 2i\left(\partial^m\bar A\partial_iA - \partial^mA\partial_i\bar A - \frac{i}{4}\partial_i\psi\sigma^m\bar\psi - \frac{i}{4}\psi\sigma^m\partial_i\bar\psi\right)v_jv_m + \\ &\qquad + i\sqrt2\big(A\bar\lambda\partial_i\bar\psi - \bar A\lambda\partial_i\psi + \partial_iA\bar\lambda\bar\psi - \partial_i\bar A\lambda\psi\big)v_j + \\ &\qquad + \frac{i}{\sqrt2}\big(\partial_i\bar A\lambda\sigma_j\bar\sigma^m\psi - \partial_iA\bar\psi\bar\sigma^m\sigma_j\bar\lambda\big)v_m\Big] + \\ &\quad + \frac{e^2}{4}\Big[2i\partial_i\bar A\partial_jAv^mv_m + (\bar A\partial_iA + A\partial_i\bar A)v_jv^mv_m - \\ &\qquad - 2\bar AAv^m\Big[v_i(\partial_jv_m + f_{jm}) + \frac{1}{4}\varepsilon_{ijmn}\lambda\sigma^n\bar\lambda\Big] + \\ &\qquad + \frac{1}{\sqrt2}\big(\bar A\psi\sigma^m\bar\sigma_j\lambda + A\bar\lambda\bar\sigma_j\sigma^m\bar\psi\big)v_iv_m\Big]\bigg\}\,,\end{aligned} \tag{5.52}$$

$$\mathcal{S}_m' = -\frac{m}{8}\int d^4x\,\vartheta^{ij}\Big\{(\psi_+\psi_- - A_+F_- - A_-F_+)f_{ij} + \\ + i\sqrt{2}(A_+\psi_- + A_-\psi_+)\sigma_j\partial_i\bar{\lambda}\Big\} + \text{c.c.} \tag{5.53}$$

Since all component fields in $\mathcal{S}'_{int_{(+)}}$ have the same lower index we have omitted it to make the formula more readable.

6. Seiberg-Witten map on euclidean space

We will be working on canonically deformed euclidean superspace, but we will continue to use lorentzian signature notation. We pass from Minkowski to euclidean space in the convenient way by substituting $x^0 \to -ix^0$. Although the formulas in section 2 still hold we should bear in mind that dotted and undotted spinors, e.g. θ and $\bar{\theta}$, are independent objects on euclidean space. This applies to chiral and antichiral fields, too.

The defining equations for the Seiberg-Witten map remain the same (4.8)–(4.10). Thus the nonchiral, nonlocal and nonsupersymmetric Seiberg-Witten maps obtained in the previous section on Minkowski space remain unchanged on euclidean space.

Let us now take the noncommutativity parameter to be selfdual ${}^*\vartheta^{ij} = \vartheta^{ij}$ and denote it by ϑ^{ij}_{SD} in what follows. In this case there exists a local, chiral and supersymmetric solution of the consistency condition (4.8) with ϑ^{ij} replaced by ϑ^{ij}_{SD}. It is

$$\Lambda' = \frac{i}{64}P^{\alpha\beta}_{SD}\bar{D}^2(D_\alpha\Lambda D_\beta V)\,, \tag{6.1}$$

where[4]

$$P^{\alpha\beta}_{SD} = \vartheta^{SD}_{ij}\varepsilon^{\alpha\mu}(\sigma^{ij})_\mu{}^\beta\,. \tag{6.2}$$

With some calculation one can show that the solution (6.1) can be written as

$$\Lambda' = \Lambda'_{\text{nonchiral}} + \Lambda'_{\text{hom}}\,, \tag{6.3}$$

where $\Lambda'_{\text{nonchiral}}$ is the nonchiral solution (5.5) with $P^{\dot{\alpha}\alpha\dot{\beta}\beta}$ replaced by

$$P^{\dot{\alpha}\alpha\dot{\beta}\beta}_{SD} = \vartheta^{SD}_{ij}\bar{\sigma}^{i\,\dot{\alpha}\alpha}\bar{\sigma}^{j\,\dot{\beta}\beta}\,. \tag{6.4}$$

The second term

$$\Lambda'_{hom} = -\frac{i}{16}P^{\alpha\beta}_{SD}D_\alpha\Lambda W_\beta \tag{6.5}$$

is a solution of the homogenous consistency condition.

The corresponding Seiberg-Witten map for the field Φ' is

$$\Phi'(\Phi,V) = -\frac{1}{16}P^{\dot{\alpha}\alpha\dot{\beta}\beta}_{SD}(\partial_{\alpha\dot{\alpha}}\Phi)(\bar{D}_{\dot{\beta}}D_\beta V) - \frac{i}{16}P^{\alpha\beta}_{SD}D_\alpha\Lambda W_\beta\,. \tag{6.6}$$

In order to determine the Seiberg-Witten map for the gauge superfield $\hat{V}$, we have first to ascertain the Seiberg-Witten map for the superfield $\bar{\hat{\Lambda}}$. On the Minkowski space this can be easily done by taking the complex conjugate of $\hat{\Lambda}$, whereas on euclidean space in we

[4] It doesn't matter if we use a general ϑ^{ij} or ϑ^{SD}_{ij} in (6.2) since $\vartheta^{SD}_{ij}\sigma^{ij} = 2\vartheta_{ij}\sigma^{ij}$ due to the selfduality of σ^{ij}.

have to solve the consistency condition for $\hat{\bar{\Lambda}}$ explicitly. In the abelian case the consistency condition for $\hat{\bar{\Lambda}}$ to first order in ϑ_{ij}^{SD} is

$$\delta_\Lambda \bar{\Sigma}'(\bar{\Sigma}, V) - \delta_\Sigma \bar{\Lambda}'(\bar{\Lambda}, V) = \vartheta_{ij}^{SD} \partial_i \bar{\Lambda} \partial_j \bar{\Sigma} \,. \tag{6.7}$$

From section 5.1 it is obvious that the only local, antichiral and supersymmetric ansatz for the Seiberg-Witten map for $\bar{\Lambda}'$ is

$$\bar{\Lambda}' = a\bar{P}_{ASD}^{\dot{\alpha}\dot{\beta}} D^2(\bar{D}_{\dot{\alpha}} \Lambda \bar{D}_{\dot{\beta}} V) \,, \tag{6.8}$$

with

$$\bar{P}_{ASD}^{\dot{\alpha}\dot{\beta}} = \vartheta_{ij}^{ASD} \varepsilon_{\dot{\beta}\dot{\mu}} (\bar{\sigma}^{ij})^{\dot{\alpha}}{}_{\dot{\mu}} \,. \tag{6.9}$$

This is however not a solution of the consistency condition (6.7). But it is obvious that it would be a solution if we take the noncommutativity parameter to be antiselfdual. Thus if the deformation parameter is selfdual we are forced to give up the antichirality of the superfields $\hat{\bar{\Lambda}}$ and $\hat{\bar{\Phi}}$. Contrariwise, if the deformation parameter is antiselfdual we are forced to give up the chirality of the superfields $\hat{\Lambda}$ and $\hat{\Phi}$.

A nonchiral Seiberg-Witten map for the field $\bar{\Lambda}'$ is

$$\bar{\Lambda}'(\bar{\Lambda}, V) = \frac{1}{16} P_{SD}^{\dot{\alpha}\alpha\dot{\beta}\beta} (\partial_{\alpha\dot{\alpha}} \bar{\Lambda})(D_\beta \bar{D}_{\dot{\beta}} V) \,. \tag{6.10}$$

It is easy to guess the Seiberg-Witten map for the field $\bar{\Phi}'$ and the gauge superfield V' which correspond to solutions (6.1) and (6.10). They are

$$\bar{\Phi}'(\bar{\Phi}, V) = \frac{1}{16} P^{\dot{\alpha}\alpha\dot{\beta}\beta} (\partial_{\alpha\dot{\alpha}} \bar{\Phi})(D_\beta \bar{D}_{\dot{\beta}} V) \,, \tag{6.11}$$

$$V'(V) = -\frac{1}{16} P_{SD}^{\dot{\alpha}\alpha\dot{\beta}\beta} (\partial_{\alpha\dot{\alpha}} V)(\bar{D}_{\dot{\beta}} D_\beta V) + \frac{i}{16} P_{SD}^{\alpha\beta} D_\alpha V W_\beta \,. \tag{6.12}$$

In the Wess-Zumino gauge the field Φ' has the usual component expansion (2.18), Λ' as given in (5.31) and V' as follows:

$$\begin{aligned} V'(x, \theta, \bar{\theta}) = & -i\bar{\theta}\bar{\chi}'(x) + \bar{\theta}\bar{\theta}\bar{M}' - \theta\sigma^m\bar{\theta} v_m'(x) + i\theta\theta\bar{\theta}\bar{\lambda}'(x) - \\ & -i\bar{\theta}\bar{\theta}\theta \left[\lambda'(x) - \frac{i}{2}\sigma^m \partial_m \bar{\chi}'(x) \right] + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta} d'(x) \,. \end{aligned} \tag{6.13}$$

The Seiberg Witten maps for the component fields are

$$\alpha'(\alpha, v) = -\frac{1}{4} \vartheta_{SD}^{ij} \partial_i \alpha v_j \,, \tag{6.14}$$

$$\eta'(\alpha, \bar{\lambda}) = -\frac{i}{4} \vartheta_{SD}^{ij} \sigma_i \bar{\lambda} \partial_j \alpha \,, \tag{6.15}$$

$$A'(A, \psi, v, \lambda) = -\frac{1}{4} \vartheta_{SD}^{ij} \left(\partial_i A v_j - \frac{1}{2\sqrt{2}} \psi \sigma^{ij} \lambda \right) , \tag{6.16}$$

$$\begin{aligned} \psi'(A, \psi, F, v, \lambda, \bar{\lambda}, d) = & -\frac{1}{4} \vartheta_{SD}^{ij} \Big(\partial_i \psi v_j + \frac{i}{\sqrt{2}} \sigma_j \bar{\lambda} \partial_i A - \frac{1}{4} \sigma^m \bar{\sigma}_j \psi f_{im}^{SD} + \\ & + \frac{i}{4} \sigma^{ij} \psi d + \frac{1}{2\sqrt{2}} \sigma^{ij} \lambda F \Big) , \end{aligned} \tag{6.17}$$

$$F^{'}(\psi,F,v,\bar\lambda) = -\frac{1}{4}\vartheta^{ij}_{SD}\left(\partial_i F v_j - \frac{i}{\sqrt{2}}\,\partial_i\left(\psi\sigma_j\bar\lambda\right) - \frac{1}{2}\,F f^{SD}_{ij}\right), \tag{6.18}$$

$$\bar\chi^{'}(v,\lambda) = \frac{i}{8}\,\vartheta^{ij}_{SD}\bar\sigma_j\lambda v_i\,, \tag{6.19}$$

$$\bar M^{'}(\lambda,\bar\lambda) = -\frac{i}{16}\,\vartheta^{ij}_{SD}\lambda\sigma^{ij}\bar\lambda\,, \tag{6.20}$$

$$v^{'}_m(v,\lambda,\bar\lambda) = \frac{1}{8}\,\vartheta^{ij}_{SD}\Big(v_i\left(2\partial_j v_m - f^{SD}_{jm} + i\eta_{jm}d\right) + i\eta_{im}\lambda\sigma_j\bar\lambda\Big), \tag{6.21}$$

$$\bar\lambda^{'}(v,\bar\lambda) = -\frac{1}{4}\vartheta^{ij}_{SD}\Big(\partial_i\bar\lambda v_j - \frac{1}{4}\bar\lambda f^{SD}_{ij} - \frac{1}{2}\,\bar\sigma^m\sigma_j\bar\lambda\partial_i v_m - \frac{1}{2}\bar\sigma_j\sigma^m\partial_m\bar\lambda v_i\Big), \tag{6.22}$$

$$\lambda^{'}(\lambda,v) = -\frac{1}{4}\,\vartheta^{ij}_{SD}\Big(\partial_i\lambda v_j - \frac{1}{4}\,\sigma^m\bar\sigma_j\lambda\left(f^{SD}_{im} + 2\partial_i v_m\right) + \frac{i}{2}\,\sigma^{ij}\lambda d\Big), \tag{6.23}$$

$$\begin{aligned} d^{'}(v,\lambda,\bar\lambda,d) = -\frac{1}{4}\,\vartheta^{ij}_{SD}\Big(&\frac{1}{2}\,\partial_i(v_j d) - \partial_i v^m\,{}^\star f_{jm} + \frac{i}{2}\,\partial^m(v_i f^{SD}_{jm}) + \\ &+ 2\lambda\sigma_j\partial_i\bar\lambda - \frac{1}{2}\,\partial_i(\lambda\sigma_j\bar\lambda)\Big)\,. \end{aligned} \tag{6.24}$$

Taking the complex conjugate of the equations (5.12) and (5.13), we get the Seiberg-Witten maps for the component fields of $\bar\Lambda^{'}$ and $\bar\Phi^{'}$. Note that by complex conjugation on Minkowski space f^{SD}_{jm} changes to f^{ASD}_{jm}.

From the component expansion of the field $\bar\Lambda^{'}$ and equation (6.21) it is evident that this solution has the same peculiar properties as the nonchiral solution on Minkowski superspace (see section 5.2).

The determination of the Seiberg-Witten maps on the canonically deformed euclidean superspace with an antiselfdual noncommutativity parameter is straight forward and will not be considered further.

7. Conclusion

On canonically deformed $N = 1$, $d = 4$ Minkowski superspace there is no Seiberg-Witten map for (anti)chiral superfields which is at the same time (anti)chiral, local and supersymmetric. We have presented Seiberg-Witten maps for superfields in each case where one of this requirements is given up and constructed the U(1) gauge theory coupled to matter in terms of classical component fields using the nonsupersymmetric Seiberg-Witten map.

The reason such a no-go theorem exists is that the chiral constraint contains derivatives, and therefore, in order to keep the constraint for more generic terms that can be generated by the Seiberg-Witten map, one needs a nonlocal projector.

On canonically deformed $N = 1$, $d = 4$ euclidean superspace we face the same incompatibility for a general noncommutativity parameter. However, if we take the noncommutativity parameter to be selfdual (antiselfdual) there exists a chiral (antichiral), local and supersymmetric Seiberg-Witten map for the chiral (antichiral) superfields and only antichirality (chirality) is broken. Thus it is not possible to construct in terms of the Seiberg-Witten map a local and supersymmetric gauge theory with both chiral and antichiral superfields on these spaces.

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References

[1] A. Konechny and A. Schwarz, *Introduction to M(atrix) theory and noncommutative geometry*, *Phys. Rept.* **360** (2002) 353 [hep-th/0012145]; *Introduction to M(atrix) theory and noncommutative geometry, II*, *Phys. Rept.* **360** (2002) 353 [hep-th/0107251].

[2] M.R. Douglas and N.A. Nekrasov, *Noncommutative field theory*, *Rev. Mod. Phys.* **73** (2001) 977 [hep-th/0106048].

[3] R.J. Szabo, *Quantum field theory on noncommutative spaces*, *Phys. Rept.* **378** (2003) 207 [hep-th/0109162].

[4] A. Connes, M.R. Douglas and A. Schwarz, *Noncommutative geometry and matrix theory: compactification on tori*, *J. High Energy Phys.* **02** (1998) 003 [hep-th/9711162].

[5] M.R. Douglas and C.M. Hull, *D-branes and the noncommutative torus*, *J. High Energy Phys.* **02** (1998) 008 [hep-th/9711165].

[6] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, *J. High Energy Phys.* **09** (1999) 032 [hep-th/9908142].

[7] J. Madore, S. Schraml, P. Schupp and J. Wess, *Gauge theory on noncommutative spaces*, *Eur. Phys. J.* **C 16** (2000) 161 [hep-th/0001203].

[8] B. Jurco, S. Schraml, P. Schupp and J. Wess, *Enveloping algebra valued gauge transformations for non-abelian gauge groups on non-commutative spaces*, *Eur. Phys. J.* **C 17** (2000) 521 [hep-th/0006246].

[9] B. Jurco, P. Schupp and J. Wess, *Nonabelian noncommutative gauge theory via noncommutative extra dimensions*, *Nucl. Phys.* **B 604** (2001) 148 [hep-th/0102129].

[10] B. Jurco, L. Moller, S. Schraml, P. Schupp and J. Wess, *Construction of non-abelian gauge theories on noncommutative spaces*, *Eur. Phys. J.* **C 21** (2001) 383 [hep-th/0104153].

[11] A. Armoni, *Comments on perturbative dynamics of non-commutative Yang-Mills theory*, *Nucl. Phys.* **B 593** (2001) 229 [hep-th/0005208].

[12] L. Bonora, M. Schnabl, M.M. Sheikh-Jabbari and A. Tomasiello, *Noncommutative* $\mathrm{SO}(N)$ *and* $\mathrm{Sp}(N)$ *gauge theories*, *Nucl. Phys.* **B 589** (2000) 461 [hep-th/0006091].

[13] I. Bars, M.M. Sheikh-Jabbari and M.A. Vasiliev, *Noncommutative* $\mathrm{o}_*(\mathrm{N})$ *and* $\mathrm{usp}_*(2\mathrm{N})$ *algebras and the corresponding gauge field theories*, *Phys. Rev.* **D 64** (2001) 086004 [hep-th/0103209].

[14] M. Chaichian, P. Prešnajder, M.M. Sheikh-Jabbari and A. Tureanu, *Noncommutative gauge field theories: a no-go theorem*, *Phys. Lett.* **B 526** (2002) 132 [hep-th/0107037].

[15] M. Hayakawa, *Perturbative analysis on infrared aspects of noncommutative QED on R^4*, *Phys. Lett.* **B 478** (2000) 394 [hep-th/9912094].

[16] M. Hayakawa, *Perturbative analysis on infrared and ultraviolet aspects of noncommutative qed on R^4*, hep-th/9912167.

[17] X. Calmet, B. Jurco, P. Schupp, J. Wess and M. Wohlgenannt, *The standard model on non-commutative space-time*, *Eur. Phys. J.* **C 23** (2002) 363 [hep-ph/0111115].

[18] P. Aschieri, B. Jurco, P. Schupp and J. Wess, *Non-commutative GUTs, standard model and C, P, T*, *Nucl. Phys.* **B 651** (2003) 45 [hep-th/0205214].

[19] C.-S. Chu and F. Zamora, *Manifest supersymmetry in non-commutative geometry*, *J. High Energy Phys.* **02** (2000) 022 [hep-th/9912153].

[20] H. Ooguri and C. Vafa, *The c-deformation of gluino and non-planar diagrams*, *Adv. Theor. Math. Phys.* **7** (2003) 53 [hep-th/0302109].

[21] H. Ooguri and C. Vafa, *Gravity induced C-deformation*, hep-th/0303063.

[22] N. Seiberg, *Noncommutative superspace, $N = 1/2$ supersymmetry, field theory and string theory*, *J. High Energy Phys.* **06** (2003) 010 [hep-th/0305248].

[23] J. de Boer, P.A. Grassi and P. van Nieuwenhuizen, *Non-commutative superspace from string theory*, *Phys. Lett.* **B 574** (2003) 98 [hep-th/0302078].

[24] S. Ferrara and M.A. Lledo, *Some aspects of deformations of supersymmetric field theories*, *J. High Energy Phys.* **05** (2000) 008 [hep-th/0002084].

[25] D. Klemm, S. Penati and L. Tamassia, *Non(anti)commutative superspace*, *Class. and Quant. Grav.* **20** (2003) 2905 [hep-th/0104190].

[26] S. Ferrara, M.A. Lledo and O. Macia, *Supersymmetry in noncommutative superspaces*, *J. High Energy Phys.* **09** (2003) 068 [hep-th/0307039].

[27] J.-H. Park, *Superfield theories and dual supermatrix models*, *J. High Energy Phys.* **09** (2003) 046 [hep-th/0307060].

[28] S. Terashima, *A note on superfields and noncommutative geometry*, *Phys. Lett.* **B 482** (2000) 276 [hep-th/0002119].

[29] S. Terashima and J.-T. Yee, *Comments on noncommutative superspace*, *J. High Energy Phys.* **12** (2003) 053 [hep-th/0306237].

[30] T. Araki, K. Ito and A. Ohtsuka, *Supersymmetric gauge theories on noncommutative superspace*, *Phys. Lett.* **B 573** (2003) 209 [hep-th/0307076].

[31] V. Putz and R. Wulkenhaar, *Seiberg-Witten map for noncommutative super Yang-Mills theory*, *Int. J. Mod. Phys.* **A 18** (2003) 3325 [hep-th/0205094].

[32] O.F. Dayi, K. Ulker and B. Yapiskan, *Duals of noncommutative supersymmetric* U(1) *gauge theory*, *J. High Energy Phys.* **10** (2003) 010 [hep-th/0309073].

[33] L.O. Chekhov and A.K. Khizhnyakov, *Gauge anomalies and the Witten-Seiberg correspondence for $N = 1$ supersymmetric theories on noncommutative spaces*, hep-th/0103048.

[34] D. Mikulović, work in progress.

[35] J. Wess and J. Bagger, *Supersymmetry and supergravity*, Princeton University Press, Princeton 1992.

[36] J. Wess and B. Zumino, *Covariant differential calculus on the quantum hyperplane*, *Nucl. Phys.* **B18** *(Proc. Suppl.)* (1991) 302.

[37] A.A. Bichl, J.M. Grimstrup, L. Popp, M. Schweda and R. Wulkenhaar, *Deformed QED via Seiberg-Witten map*, `hep-th/0102103`.

[38] S. Paban, S. Sethi and M. Stern, *Non-commutativity and supersymmetry*, *J. High Energy Phys.* **03** (2002) 012 [`hep-th/0201259`].

[39] W. Siegel and S.J. Gates Jr., *Superfield supergravity*, *Nucl. Phys.* **B 147** (1979) 77.

[40] S.J. Gates Jr., K.S. Stelle and P.C. West, *Algebraic origins of superspace constraints in supergravity*, *Nucl. Phys.* **B 169** (1980) 347.

[41] D. Berenstein and S.-J. Rey, *Wilsonian proof for renormalizability of $N = 1/2$ supersymmetric field theories*, *Phys. Rev.* **D 68** (2003) 121701 [`hep-th/0308049`].